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A CLASS OF SPATIAL CENTRAL CONFIGURATIONS IN NEWTONIAN NINE-BODY PROBLEMS

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Abstract. In this paper, we obtain the necessary and sufficient conditions for the existence of a class of spatial Newtonian nine-body central configurations, where eight masses are at the vertices of a rectangular prism and the ninth mass is at the geometry center of the rectangular prism.

Keywords: Newtonian nine-body problems, spatial central configurations, squares, rectangles, rectangular prism, cube.

AMS Subject Classification: 70F10, 70F15.

1. Introduction and main result

Newtonian n-body problems given in [12, 25] concern the motion of n point masses $m_j > 0$ with positions $q_j \in \mathbb{R}^3$ for $j = 1, 2, \ldots, n$. The motion is governed by Newton's law:

$$m_j \ddot{q}_j = \frac{\partial U(q)}{\partial q_j},$$

where $U(q)$ is the potential function as following:

$$U(q) = \sum_{1 \leq i \neq j \leq n} \frac{m_i m_j}{|q_i - q_j|}.$$

One of the main topics in Newtonian n-body problems is central configuration, and the definition of central configuration is as follows [12].

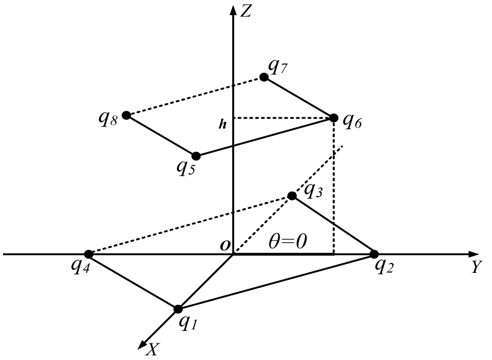

Figure 1. Planar N-body problem.

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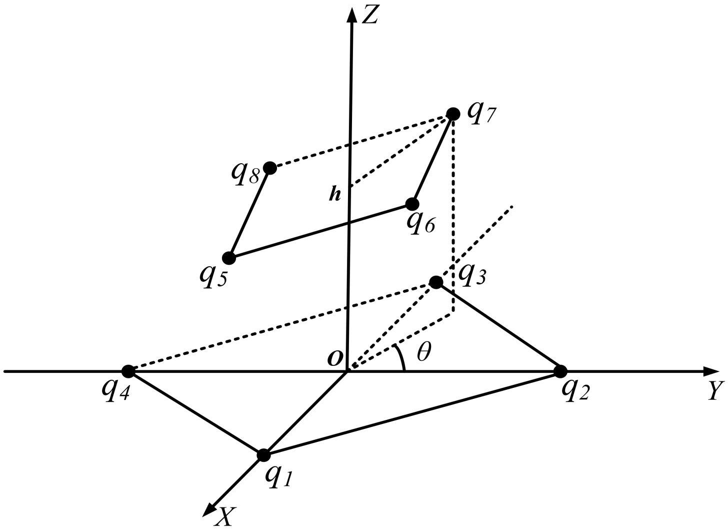

Figure 2. Eight-body configurations formed by two parallel squares with $\theta \neq 0$.

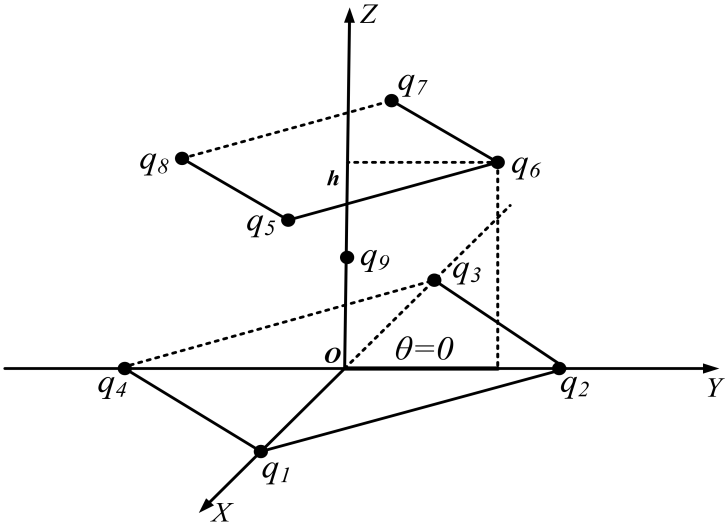

Figure 3. Nine-body configurations formed by two parallel squares with $\theta = 0$.

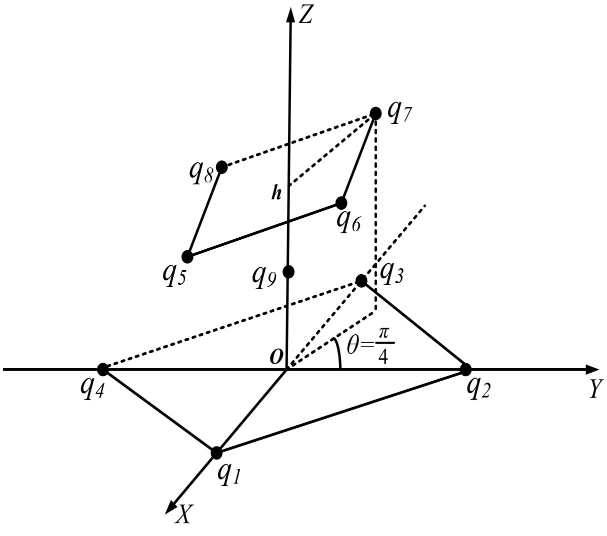

Figure 4. Nine-body configurations formed by two parallel squares with $\theta = \pi/4$.

Definition 1.1. *A configuration* $q = (q_1, q_2, \dots, q_n) \in \Omega$ *is called a central configuration for masses* $m = (m_1, m_2, \dots, m_n)$ *if there exists a constant* $\lambda \in \mathbb{R}$ *such that*

$$\sum_{j=1,\ j\neq i}^{n} \frac{m_i m_j (q_j - q_i)}{|q_j - q_i|^3} = -\lambda m_i (q_i - q_0), \quad i = 1, 2, \dots, n, \tag{1}$$

where the configuration space $\Omega = \left\{(q_1, \cdots, q_n) \in (\mathbb{R}^3)^n : q_i \neq q_j \text{ for } i \neq j\right\}$, *and the center of mass is defined as* $q_0 = \sum\limits_{j=1}^{n} m_j q_j / M$ *and* $M = \sum\limits_{j=1}^{n} m_j$.

The study of central configuration dates back to the 18th century, more precisely, in 1767, Euler [8] discover the well-known collinear central configuration for the case of $n = 3$. For the planar non-collinear central configuration with $n = 3$, there is only one kind of central configuration, that is Lagrange equilateral-triangle central configuration [11]. For $n \geq 4$, if n equal masses are located at the vertices of a regular n-polygon, then the n masses form a central configuration [19], and for the spatial central configuration with $n = 4$, there is only one kind of central configuration, which is called the regular tetrahedron central configuration [1, 20]. For the spatial case of $n = 5$, the authors of the paper [5] discovered the spatial central configurations with four sequential equilateral edges containing all five vertices. When $n = 6$, in [15] the authors showed the existence for spatial central configurations with four masses are at the vertices of a regular tetrahedron and the other two masses are on a line connecting one vertex of the tetrahedron with the center of the opposite face. For the spatial case with $n = 7$, the authors of [16] discovered the existence of spatial central configurations, where four masses are at the vertices of a regular tetrahedron and the other three masses are symmetrically located at the vertices of an equilateral triangle in the exterior of the regular tetrahedron, and when $n = 8$, in [18] the authors showed the existence for the spatial central configurations formed by two parallel squares with a twist angle $\theta = 0$ and a distance $h > 0$ (see Fig.1). In [27], the authors studied the necessary and sufficient conditions for twisted angles in central configurations formed by two twisted regular polygons (see Figs.1, 2), and later, they found that two stacked regular polygons forming a symmetrical central configuration have the same shape [28]. In [4, 21], the authors extended the configurations studied in [18, 27, 28] by adding a mass and established the existence of the spatial central configurations for Newtonian nine-body problems with $\theta = 0$ and $\theta = \pi/4$, respectively (see Figs.3, 4). For more details in this direction, one can refer to [2, 3, 6, 7, 13, 14, 17, 22, 23, 24, 26].

Note that in [4, 21], the authors investigated spatial central configurations of Newtonian nine-body problems where the five masses are at the vertices of a square-base pyramid while the remaining four masses are at the vertices of a square, they demonstrated that the masses at the vertices of each square must be equal. We also note that in a co-spherical central configuration, if adding one more celestial body still results in a central configuration, then the newly added body must lie at the center of the sphere [10, 29]. Now in this paper, we focus on a class of spatial central configurations consisting of eight masses at the vertices of a rectangular prism and the ninth mass at the geometric center of the rectangular prism (see Fig.5), and we make no prior assumption that the first eight bodies form a central configuration. Since two central configurations are said to be equivalent if one can be transformed to the other by a translation, a scalar multiplication and a rotation [12], without loss of generality, we can assume that the width, length and height of the rectangular prism are 1, a, and h, respectively, i.e.,

$$\begin{cases} q_1 = (1, 0, 0), \ q_2 = (1, a, 0), \ q_3 = (0, a, 0), \\ q_4 = (0, 0, 0), \ q_5 = (1, 0, h), \ q_6 = (1, a, h), \\ q_7 = (0, a, h), \ q_8 = (0, 0, h), \ q_9 = (\frac{1}{2}, \frac{a}{2}, \frac{h}{2}), \end{cases} \tag{2}$$

where $a > 0$ and $h > 0$. Then, our main result is described as follows.

Theorem 1.1. *For the spatial Newtonian nine-body problems with masses* $m_1, m_2, \ldots, m_9 \in \mathbb{R}^+$ *and corresponding positions* $q_1, q_2, \ldots, q_9 \in \mathbb{R}^3$, *if the positions of the nine masses are defined as in (2), then the nine masses form a central configuration if and only if* $m_1 = m_2 = \cdots = m_8$ *and the rectangular prism is a cube.*

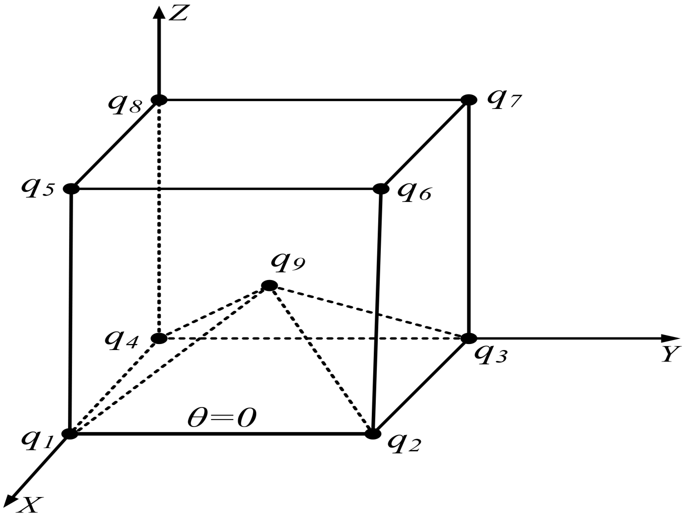

Figure 5. Nine-body configurations formed by a rectangular prism and the ninth mass is at the geometry center of the rectangular prism.

2. Proof of the necessary condition of Theorem 1.1.

For the convenience to discuss, we use $D_{i,j}$ to denote the mutual distance between q_i and q_j with $i, j \in \{1, 2, \ldots, 9\}$. Moreover, let $\overrightarrow{e_1} = (1, 0, 0)$, $\overrightarrow{e_2} = (0, 1, 0)$ and $\overrightarrow{e_3} = (0, 0, 1)$.

We divide the proof of the necessary condition into two parts.

Part 1. We prove that $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = m_8$, and we divide the proof into three steps.

Step 1. We prove that $m_1 = m_3$, $m_2 = m_4$, $m_5 = m_7$ and $m_6 = m_8$.

Since $q_1, q_2, \ldots, q_9$ form a central configuration, by Definition 1.1, there exists a constant $\lambda \in \mathbb{R}$ such that

$$\sum_{j=1,\ j\neq i}^{9} \frac{m_i m_j (q_j - q_i)}{D_{ij}^3} = -\lambda m_i (q_i - q_0), \quad i = 1, 2, \ldots, 9, \tag{3}$$

where $q_0 = \sum\limits_{j=1}^{9} m_j q_j / M$ and $M = \sum\limits_{j=1}^{9} m_j$.

When $i = 5$ and $i = 7$, we multiply both sides of Eqn.(3) by $\overrightarrow{e_3}$. Then, using Eqn.(2), we have

$$\begin{cases} \dfrac{m_1}{D_{51}^3} + \dfrac{m_2}{D_{52}^3} + \dfrac{m_3}{D_{53}^3} + \dfrac{m_4}{D_{54}^3} + \dfrac{\frac{1}{2}m_9}{D_{59}^3} = \lambda\Big[1 - \dfrac{1}{M}(m_5 + m_6 + m_7 + m_8 + \dfrac{1}{2}m_9)\Big], \\ \dfrac{m_1}{D_{71}^3} + \dfrac{m_2}{D_{72}^3} + \dfrac{m_3}{D_{73}^3} + \dfrac{m_4}{D_{74}^3} + \dfrac{\frac{1}{2}m_9}{D_{79}^3} = \lambda\Big[1 - \dfrac{1}{M}(m_5 + m_6 + m_7 + m_8 + \dfrac{1}{2}m_9)\Big], \end{cases} \tag{4}$$

where

$$\begin{cases} D_{51} = D_{73} = h,\ D_{52} = D_{74} = \sqrt{a^2+h^2},\ D_{53} = D_{71} = \sqrt{1+a^2+h^2}, \\ D_{54} = D_{72} = \sqrt{1+h^2},\ D_{59} = D_{79} = \sqrt{\frac{1}{4}+\frac{a^2}{4}+\frac{h^2}{4}}. \end{cases}$$

Thus, in Eqn.(4), using the first part to minus the second part, we have

$$\begin{aligned} & [\frac{m_1}{D_{51}^3} + \frac{m_2}{D_{52}^3} + \frac{m_3}{D_{53}^3} + \frac{m_4}{D_{54}^3} + \frac{\frac{1}{2}m_9}{D_{59}^3}] - [\frac{m_1}{D_{71}^3} + \frac{m_2}{D_{72}^3} + \frac{m_3}{D_{73}^3} + \frac{m_4}{D_{74}^3} + \frac{\frac{1}{2}m_9}{D_{79}^3}] \\ = & [\frac{m_1}{D_{51}^3} + \frac{m_2}{D_{52}^3} + \frac{m_3}{D_{53}^3} + \frac{m_4}{D_{54}^3}] - [\frac{m_1}{D_{71}^3} + \frac{m_2}{D_{72}^3} + \frac{m_3}{D_{73}^3} + \frac{m_4}{D_{74}^3}] \\ = & \frac{1}{D_{51}^3}(m_1 - m_3) + \frac{1}{D_{52}^3}(m_2 - m_4) + \frac{1}{D_{53}^3}(m_3 - m_1) + \frac{1}{D_{54}^3}(m_4 - m_2) \\ = & (\frac{1}{D_{51}^3} - \frac{1}{D_{53}^3})(m_1 - m_3) + (\frac{1}{D_{52}^3} - \frac{1}{D_{54}^3})(m_2 - m_4) = 0, \end{aligned}$$

i.e.,

$$\Big[\frac{1}{h^3} - \frac{1}{(\sqrt{1+a^2+h^2})^3}\Big](m_1 - m_3) + \Big[\frac{1}{(\sqrt{a^2+h^2})^3} - \frac{1}{(\sqrt{1+h^2})^3}\Big](m_2 - m_4) = 0. \tag{5}$$

Similar to obtain Eqn.(5), when $i = 6$ and $i = 8$, multiplying both sides of Eqn.(3) by $\overrightarrow{e_3}$, we obtain

$$\begin{cases} \frac{m_1}{D_{61}^3} + \frac{m_2}{D_{62}^3} + \frac{m_3}{D_{63}^3} + \frac{m_4}{D_{64}^3} + \frac{\frac{1}{2}m_9}{D_{69}^3} = \lambda\Big[1 - \frac{1}{M}(m_5 + m_6 + m_7 + m_8 + \frac{1}{2}m_9)\Big], \\ \frac{m_1}{D_{81}^3} + \frac{m_2}{D_{82}^3} + \frac{m_3}{D_{83}^3} + \frac{m_4}{D_{84}^3} + \frac{\frac{1}{2}m_9}{D_{89}^3} = \lambda\Big[1 - \frac{1}{M}(m_5 + m_6 + m_7 + m_8 + \frac{1}{2}m_9)\Big], \end{cases}$$

where

$$\begin{cases} D_{61} = D_{83} = \sqrt{a^2+h^2},\ D_{62} = D_{84} = h,\ D_{63} = D_{81} = \sqrt{1+h^2}, \\ D_{64} = D_{82} = \sqrt{1+a^2+h^2},\ D_{69} = D_{89} = \sqrt{\frac{1}{4}+\frac{a^2}{4}+\frac{h^2}{4}}. \end{cases}$$

Then,

$$(\frac{1}{D_{61}^3} - \frac{1}{D_{63}^3})(m_1 - m_3) + (\frac{1}{D_{62}^3} - \frac{1}{D_{64}^3})(m_2 - m_4) = 0,$$

i.e.,

$$\Big[\frac{1}{(\sqrt{a^2+h^2})^3} - \frac{1}{(\sqrt{1+h^2})^3}\Big](m_1 - m_3) + \Big[\frac{1}{h^3} - \frac{1}{(\sqrt{1+a^2+h^2})^3}\Big](m_2 - m_4) = 0. \tag{6}$$

Let matrix A be defined as follows.

$$A = \begin{pmatrix} \frac{1}{h^3} - \frac{1}{(\sqrt{1+a^2+h^2})^3} & \frac{1}{(\sqrt{a^2+h^2})^3} - \frac{1}{(\sqrt{1+h^2})^3} \\ \frac{1}{(\sqrt{a^2+h^2})^3} - \frac{1}{(\sqrt{1+h^2})^3} & \frac{1}{h^3} - \frac{1}{(\sqrt{1+a^2+h^2})^3} \end{pmatrix}.$$

Then, it follows from Eqns.(5) and (6) that

$$A\begin{pmatrix} m_1 - m_3 \\ m_2 - m_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the determinant of matrix A is following:

$$|A| = \begin{vmatrix} \dfrac{1}{h^3} - \dfrac{1}{(\sqrt{1+a^2+h^2})^3} & \dfrac{1}{(\sqrt{a^2+h^2})^3} - \dfrac{1}{(\sqrt{1+h^2})^3} \\ \dfrac{1}{(\sqrt{a^2+h^2})^3} - \dfrac{1}{(\sqrt{1+h^2})^3} & \dfrac{1}{h^3} - \dfrac{1}{(\sqrt{1+a^2+h^2})^3} \end{vmatrix}$$

$$= \Big[\frac{1}{h^3} - \frac{1}{(\sqrt{1+a^2+h^2})^3}\Big]^2 - \Big[\frac{1}{(\sqrt{a^2+h^2})^3} - \frac{1}{(\sqrt{1+h^2})^3}\Big]^2$$

$$= \Big[\frac{1}{h^3} - \frac{1}{(\sqrt{1+a^2+h^2})^3} + \frac{1}{(\sqrt{a^2+h^2})^3} - \frac{1}{(\sqrt{1+h^2})^3}\Big]\Big[\frac{1}{h^3}$$

$$-\frac{1}{(\sqrt{1+a^2+h^2})^3} - \frac{1}{(\sqrt{a^2+h^2})^3} + \frac{1}{(\sqrt{1+h^2})^3}\Big].$$

Note that $a > 0$ and $h > 0$, then $|A| > 0$, hence $m_1 - m_3 = 0$ and $m_2 - m_4 = 0$, which implies that $m_1 = m_3$ and $m_2 = m_4$. Following the same procedure that established $m_1 = m_3$ and $m_2 = m_4$, we set $i = 1, 3$ and $i = 2, 4$ into Eqn.(3), and then multiplying both sides of Eqn.(3) by $\overrightarrow{e_3}$, it can be shown that $m_5 = m_7$ and $m_6 = m_8$.

Step 2. We prove that $m_1 = m_2 = m_3 = m_4$ and $m_5 = m_6 = m_7 = m_8$.

For $i = 1$ and $i = 2$, by multiplying Eqn.(3) by $\overrightarrow{e_1}$, it yields that

$$\begin{cases} \dfrac{m_3}{D_{13}^3} + \dfrac{m_4}{D_{14}^3} + \dfrac{m_7}{D_{17}^3} + \dfrac{m_8}{D_{18}^3} + \dfrac{\frac{1}{2}m_9}{D_{19}^3} = \lambda\Big[1 - \dfrac{1}{M}(m_1 + m_2 + m_5 + m_6 + \dfrac{1}{2}m_9)\Big], \\ \dfrac{m_3}{D_{23}^3} + \dfrac{m_4}{D_{24}^3} + \dfrac{m_7}{D_{27}^3} + \dfrac{m_8}{D_{28}^3} + \dfrac{\frac{1}{2}m_9}{D_{29}^3} = \lambda\Big[1 - \dfrac{1}{M}(m_1 + m_2 + m_5 + m_6 + \dfrac{1}{2}m_9)\Big], \end{cases} \tag{7}$$

where

$$\begin{cases} D_{13} = D_{24} = \sqrt{1+a^2},\ D_{14} = D_{23} = 1,\ D_{17} = D_{28} = \sqrt{1+a^2+h^2}, \\ D_{18} = D_{27} = \sqrt{1+h^2},\ D_{19} = D_{29} = \sqrt{\frac{1}{4} + \frac{a^2}{4} + \frac{h^2}{4}}. \end{cases}$$

In Eqn.(7), by subtracting the second part from the first part, we obtain

$$(\frac{1}{D_{13}^3} - \frac{1}{D_{14}^3})(m_3 - m_4) + (\frac{1}{D_{17}^3} - \frac{1}{D_{18}^3})(m_7 - m_8) = 0,$$

i.e.,

$$\Big[\frac{1}{(\sqrt{1+a^2})^3} - 1\Big](m_3 - m_4) + \Big[\frac{1}{(\sqrt{1+a^2+h^2})^3} - \frac{1}{(\sqrt{1+h^2})^3}\Big](m_7 - m_8) = 0. \tag{8}$$

Similarly, for $i = 5$ and $i = 6$, by multiplying the Eqn.(3) by $\overrightarrow{e_1}$, we obtain

$$\begin{cases} \dfrac{m_3}{D_{53}^3} + \dfrac{m_4}{D_{54}^3} + \dfrac{m_7}{D_{57}^3} + \dfrac{m_8}{D_{58}^3} + \dfrac{\frac{1}{2}m_9}{D_{59}^3} = \lambda\Big[1 - \dfrac{1}{M}(m_1 + m_2 + m_5 + m_6 + \dfrac{1}{2}m_9)\Big], \\ \dfrac{m_3}{D_{63}^3} + \dfrac{m_4}{D_{64}^3} + \dfrac{m_7}{D_{67}^3} + \dfrac{m_8}{D_{68}^3} + \dfrac{\frac{1}{2}m_9}{D_{69}^3} = \lambda\Big[1 - \dfrac{1}{M}(m_1 + m_2 + m_5 + m_6 + \dfrac{1}{2}m_9)\Big], \end{cases} \tag{9}$$

where

$$\begin{cases} D_{53} = D_{64} = \sqrt{1+a^2+h^2},\ D_{54} = D_{63} = \sqrt{1+h^2},\ D_{57} = D_{68} = \sqrt{1+a^2}, \\ D_{58} = D_{67} = 1,\ D_{59} = D_{69} = \sqrt{\frac{1}{4} + \frac{a^2}{4} + \frac{h^2}{4}}. \end{cases}$$

In Eqn.(9), using the first part to minus the second part, we have

$$(\frac{1}{D_{53}^3}-\frac{1}{D_{54}^3})(m_3-m_4)+(\frac{1}{D_{57}^3}-\frac{1}{D_{58}^3})(m_7-m_8)=0,$$

i.e.,

$$\Big[\frac{1}{(\sqrt{1+a^2+h^2})^3}-\frac{1}{(\sqrt{1+h^2})^3}\Big](m_3-m_4)+\Big[\frac{1}{(\sqrt{1+a^2})^3}-1\Big](m_7-m_8)=0. \tag{10}$$

Denote the determinant of the coefficients of the system consisting of Eqns.(8) and (10) be $|\overline{A}|$, then

$$|\overline{A}|=\begin{vmatrix} \frac{1}{(\sqrt{1+a^2})^3}-1 & \frac{1}{(\sqrt{1+a^2+h^2})^3}-\frac{1}{(\sqrt{1+h^2})^3} \\ \frac{1}{(\sqrt{1+a^2+h^2})^3}-\frac{1}{(\sqrt{1+h^2})^3} & \frac{1}{(\sqrt{1+a^2})^3}-1 \end{vmatrix}$$

$$=\Big[\frac{1}{(\sqrt{1+a^2})^3}-1\Big]^2-\Big[\frac{1}{(\sqrt{1+a^2+h^2})^3}-\frac{1}{(\sqrt{1+h^2})^3}\Big]^2$$

$$=\Big[\frac{1}{(\sqrt{1+a^2})^3}-1+\frac{1}{(\sqrt{1+a^2+h^2})^3}-\frac{1}{(\sqrt{1+h^2})^3}\Big]\Big[\frac{1}{(\sqrt{1+a^2})^3}-1$$

$$-\frac{1}{(\sqrt{1+a^2+h^2})^3}+\frac{1}{(\sqrt{1+h^2})^3}\Big].$$

Thanks to $a>0$ and $h>0$, so

$$\frac{1}{(\sqrt{1+a^2})^3}-1+\frac{1}{(\sqrt{1+a^2+h^2})^3}-\frac{1}{(\sqrt{1+h^2})^3}<0.$$

Next, we prove $|\overline{A}|\neq 0$ by contradiction argument, and we assume that $|\overline{A}|=0$, then

$$\frac{1}{(\sqrt{1+a^2})^3}-1-\frac{1}{(\sqrt{1+a^2+h^2})^3}+\frac{1}{(\sqrt{1+h^2})^3}=0,$$

i.e.,

$$\frac{1}{(\sqrt{1+a^2})^3}-1=\frac{1}{(\sqrt{1+a^2+h^2})^3}-\frac{1}{(\sqrt{1+h^2})^3}. \tag{11}$$

Let $f_1(h)=1/(\sqrt{1+a^2})^3-1$ and $f_2(h)=1/(\sqrt{1+a^2+h^2})^3-1/(\sqrt{1+h^2})^3$. Note that for Eqn.(11) to hold, it requires that the graphs of $f_1(h)$ and $f_2(h)$ intersect. On the other hand, the derivative of $f_2(h)$ is given by:

$$f_2'(h)=3h\left[(1+h^2)^{-\frac{5}{2}}-(1+a^2+h^2)^{-\frac{5}{2}}\right],$$

then it follows from $a>0$ and $h>0$ that $f_2'(h)>0$. Therefore, $f_2(h)$ is monotonically increasing on $(0,+\infty)$. Moreover, note that as $h\to 0$, $f_2(h)\to 1/(\sqrt{1+a^2})^3-1$, thus using $h\neq 0$, we have $f_1(h)$ and $f_2(h)$ do not intersect, which implies that Eqn.(11) does not hold, and it is impossible. Consequently, the determinant $|\overline{A}|\neq 0$, and then the homogeneous linear system for (m_3-m_4) and (m_7-m_8) admits only the trivial solution, i.e., $m_3=m_4$ and $m_7=m_8$. Combining **Step 1**, we conclude that $m_1=m_2=m_3=m_4$ and $m_5=m_6=m_7=m_8$.

Step 3. We prove that $m_1=m_2=m_3=m_4=m_5=m_6=m_7=m_8$.

By virtue of **Step 2**, we can assume that $m_1 = m_2 = m_3 = m_4 = \alpha$ and $m_5 = m_6 = m_7 = m_8 = \beta$. For $i = 1$ and $i = 5$, we multiply Eqn.(3) by $\overrightarrow{e_1}$, then

$$\begin{cases} (\frac{1}{D_{13}^3} + \frac{1}{D_{14}^3})\alpha + (\frac{1}{D_{17}^3} + \frac{1}{D_{18}^3})\beta + \frac{\frac{1}{2}m_9}{D_{19}^3} = \lambda\Big[1 - \frac{1}{M}(2\alpha + 2\beta + \frac{1}{2}m_9)\Big], \\ (\frac{1}{D_{53}^3} + \frac{1}{D_{54}^3})\alpha + (\frac{1}{D_{57}^3} + \frac{1}{D_{58}^3})\beta + \frac{\frac{1}{2}m_9}{D_{59}^3} = \lambda\Big[1 - \frac{1}{M}(2\alpha + 2\beta + \frac{1}{2}m_9)\Big], \end{cases} \tag{12}$$

where

$$\begin{cases} D_{13} = D_{57} = \sqrt{1+a^2},\ D_{14} = D_{58} = 1,\ D_{17} = D_{53} = \sqrt{1+a^2+h^2}, \\ D_{18} = D_{54} = \sqrt{1+h^2},\ D_{19} = D_{59} = \sqrt{\frac{1}{4} + \frac{a^2}{4} + \frac{h^2}{4}}. \end{cases}$$

In Eqn.(12), using the first part to minus the second part, there is

$$(\frac{1}{D_{13}^3} + \frac{1}{D_{14}^3} - \frac{1}{D_{53}^3} - \frac{1}{D_{54}^3})(\alpha - \beta) = 0,$$

i.e.,

$$\Big[\frac{1}{(\sqrt{1+a^2})^3} + 1 - \frac{1}{(\sqrt{1+a^2+h^2})^3} - \frac{1}{(\sqrt{1+h^2})^3}\Big](\alpha - \beta) = 0.$$

With the aid of

$$\frac{1}{(\sqrt{1+a^2})^3} + 1 - \frac{1}{(\sqrt{1+a^2+h^2})^3} - \frac{1}{(\sqrt{1+h^2})^3} > 0,$$

we obtain $\alpha = \beta$. Therefore, $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = m_8$.

Part 2. We prove that the rectangular prism must be a cube.

Firstly, we prove that $a = 1$. Let $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = m_8 =: \alpha > 0$. For $i = 2$, we multiply Eqn.(3) by $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ respectively, which yields

$$\begin{cases} (\frac{1}{D_{23}^3} + \frac{1}{D_{24}^3} + \frac{1}{D_{27}^3} + \frac{1}{D_{28}^3})\alpha + \frac{\frac{1}{2}m_9}{D_{29}^3} = \lambda\Big[1 - \frac{1}{M}(4\alpha + \frac{1}{2}m_9)\Big], \\ (\frac{1}{D_{21}^3} + \frac{1}{D_{24}^3} + \frac{1}{D_{25}^3} + \frac{1}{D_{28}^3})\alpha + \frac{\frac{1}{2}m_9}{D_{29}^3} = \lambda\Big[1 - \frac{1}{M}(4\alpha + \frac{1}{2}m_9)\Big], \end{cases} \tag{13}$$

where

$$\begin{cases} D_{23} = 1,\ D_{24} = \sqrt{1+a^2},\ D_{27} = \sqrt{1+h^2},\ D_{28} = \sqrt{1+a^2+h^2}, \\ D_{29} = \sqrt{\frac{1}{4} + \frac{a^2}{4} + \frac{h^2}{4}},\ D_{21} = a,\ D_{25} = \sqrt{a^2+h^2}. \end{cases}$$

In Eqn.(13), using the first part to minus the second part, we obtain

$$(\frac{1}{D_{23}^3} - \frac{1}{D_{21}^3} + \frac{1}{D_{27}^3} - \frac{1}{D_{25}^3})\alpha = 0,$$

i.e.,

$$\Big[1 - \frac{1}{a^3} + \frac{1}{(\sqrt{1+h^2})^3} - \frac{1}{(\sqrt{a^2+h^2})^3}\Big]\alpha = 0. \tag{14}$$

Note that $\alpha > 0$, we have the following conclusions:

$$\begin{cases} \text{if } a > 1, \text{ then } 1 - \dfrac{1}{a^3} + \dfrac{1}{(\sqrt{1+h^2})^3} - \dfrac{1}{(\sqrt{a^2+h^2})^3} > 0, \text{ which implies that (14) does not hold;} \\ \text{if } 0 < a < 1, \text{ then } 1 - \dfrac{1}{a^3} + \dfrac{1}{(\sqrt{1+h^2})^3} - \dfrac{1}{(\sqrt{a^2+h^2})^3} < 0, \text{ which implies that (14) does not hold.} \end{cases}$$

Hence, $a = 1$.

Secondly, we prove that $h = 1$.

With all masses set equal, i.e., $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = m_8 = \alpha$, we substitute $i = 1$ into Eqn.(3). Multiplying the resulting expression by $\overrightarrow{e_2}$ and $\overrightarrow{e_3}$ respectively yields

$$\begin{cases} (\dfrac{1}{D_{12}^3} + \dfrac{1}{D_{13}^3} + \dfrac{1}{D_{16}^3} + \dfrac{1}{D_{17}^3})\alpha + \dfrac{\frac{1}{2}m_9}{D_{19}^3} = \dfrac{\lambda}{M}(4\alpha + \dfrac{1}{2}m_9), \\ (\dfrac{1}{D_{15}^3} + \dfrac{1}{D_{16}^3} + \dfrac{1}{D_{17}^3} + \dfrac{1}{D_{18}^3})\alpha + \dfrac{\frac{1}{2}m_9}{D_{19}^3} = \dfrac{\lambda}{M}(4\alpha + \dfrac{1}{2}m_9). \end{cases} \tag{15}$$

By subtracting the second part of Eqn.(15) from the first part of Eqn.(15), and then using $a = 1$, we obtain

$$(\frac{1}{D_{12}^3} + \frac{1}{D_{13}^3} - \frac{1}{D_{15}^3} - \frac{1}{D_{18}^3})\alpha = 0, \tag{16}$$

where

$$D_{12} = 1, \ D_{13} = \sqrt{2}, \ D_{15} = h, \ D_{18} = \sqrt{1+h^2}. \tag{17}$$

Inserting $\alpha > 0$ and (17) into (16), we have

$$1 + \frac{1}{(\sqrt{2})^3} - \frac{1}{h^3} - \frac{1}{(\sqrt{1+h^2})^3} = 0,$$

which implies that

$$\frac{1}{h^3} + \frac{1}{(\sqrt{1+h^2})^3} = 1 + \frac{1}{(\sqrt{2})^3}.$$

Let $f_3(h) = 1/h^3 + 1/(\sqrt{1+h^2})^3$ and $f_4(h) \equiv 1 + 1/(\sqrt{2})^3$. Since

$$f_3'(h) = -3h^{-4} - 3h(1+h^2)^{-\frac{5}{2}} < 0.$$

Hence, $f_3'(h)$ is monotonically decreasing on $(0, +\infty)$. Moreover, when $h = 1$, we have $f_3(1) = 1 + 1/(\sqrt{2})^3$, which implies that $h = 1$.

Inserting $a = 1$ and $h = 1$ into (2), we deduce that the rectangular prism is a cube. □

3. Proof of the sufficient condition of Theorem 1.1

In order to prove that the nine masses form a central configuration, it suffices to show that Eqn.(1) holds. In fact, note that the rectangular prism is a cube, then in (2), we have $a = 1$ and $h = 1$. Combining $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = m_8 =: \beta > 0$, then by direct computation, Eqn.(1) is equivalent to

$$\begin{cases} (\dfrac{1}{D_{13}^3} + \dfrac{1}{D_{14}^3} + \dfrac{1}{D_{17}^3} + \dfrac{1}{D_{18}^3})\beta + \dfrac{\frac{1}{2}m_9}{D_{19}^3} = \lambda\Big[1 - \dfrac{1}{M}(4\beta + \dfrac{1}{2}m_9)\Big], \\ (\dfrac{1}{D_{12}^3} + \dfrac{1}{D_{13}^3} + \dfrac{1}{D_{16}^3} + \dfrac{1}{D_{17}^3})\beta + \dfrac{\frac{1}{2}m_9}{D_{19}^3} = \dfrac{\lambda}{M}(4\beta + \dfrac{1}{2}m_9), \end{cases} \tag{18}$$

where

$$D_{12} = D_{14} = 1, D_{13} = D_{18} = D_{16} = \sqrt{2},\ D_{17} = \sqrt{3},\ D_{19} = \frac{\sqrt{3}}{2} \text{ and } M = 8\beta + m_9. \tag{19}$$

By the lines 1–8 of p.109 of [12], we have $\lambda > 0$. Thus, using $M = 8\beta + m_9$ and $\lambda > 0$, it follows that

$$\lambda\Big[1 - \frac{1}{M}(4\beta + \frac{1}{2}m_9)\Big] = \frac{\lambda}{M}(4\beta + \frac{1}{2}m_9) > 0.$$

Combining (19), we arrive at the conclusion that Eqn.(18) is equivalent to

$$(1 + \frac{1}{(\sqrt{2})^3} + \frac{1}{(\sqrt{2})^3} + \frac{1}{(\sqrt{3})^3})\beta + \frac{\frac{1}{2}m_9}{(\sqrt{3}/2)^3} = \frac{\lambda}{8\beta + m_9}(4\beta + \frac{1}{2}m_9) = \frac{\lambda}{2}.$$

Hence, there exists a constant $\lambda = (3/2 + 1/(3\sqrt{3}))\beta + 8m_9/(3\sqrt{3}) > 0$ such that Eqn.(1) holds. This guarantees the existence of the spatial central configurations.

4. Conclusions

In planar and spatial central configurations of Newtonian five-body problems where four masses are located at the vertices of a square, these four masses are necessarily equal [2, 9], and for spatial central configurations of Newtonian nine-body problems with eight masses positioned at the vertices of two parallel squares, the authors proved that the masses at the vertices of each separate square are equal [4, 21]. Now, we obtain the necessary and sufficient conditions for the existence of spatial central configurations consisting of eight masses positioned at the vertices of a rectangular prism and the ninth mass placed at the geometric center of the rectangular prism, specially, without assuming that the first eight masses form a central configuration beforehand, we demonstrate that in these spatial nine-body central configurations, the first eight masses are all equal and the rectangular prism must be a cube.

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